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RESUMEN

El objetivo de este artículo es el estudio de formación de redes en un modelo en el que los agentes tienen distinto poder de negociación respecto al reparto de los costes de los vínculos. Específicamente, se asume que el poder de negociación de un agente depende del hecho que este agente sea quien envía o recibe la propuesta de formar el vínculo. Se estudia este problema en un contexto en el que, cuando dos agentes establecen un vínculo (con un coste asociado), juegan un juego de coordinación bilateral. En este sentido, se amplían los contextos considerados por Meléndez-Jiménez (2008), que asume poder de negociación simétrico, y por Goyal y Vega-Redondo (2005), donde quien propone el vínculo asume unilateralmente el coste. Se demuestra que el contexto estudiado presenta un conjunto de equilibrios más variado, que incluye la posibilidad de coexistencia de acciones diferentes, tanto en componentes separados como dentro de un mismo componente de la red.

PALABRAS CLAVE: Coordinación, solución de negociación de Nash generalizada, redes.

CÓDIGOS JEL: C72; C78; D85

ABSTRACT

The aim of this paper is to study network formation in a model in which agents have different bargaining powers concerning the sharing of link costs. Specifically, we assume that the bargaining power of an agent depends on whether this agent is the proposer of the link or she receives the link proposal. We study this problem for a context in which, whenever two agents form a (costly) link, they play a bilateral coordination game. Hence, we broaden the frameworks considered in Meléndez-Jiménez (2008), that assumes symmetric bargaining power, and in Goyal and Vega-Redondo (2005), in which the proposer of the link unilaterally meets the cost. We show that our framework presents a richer equilibrium set, which includes the possibility of coexistence of different actions, both in separate components and within the same component of the network.

KEY WORDS: Coordination, generalized Nash bargaining solution, networks.
1. INTRODUCTION

In recent years, there has been a growing attention to the literature on social networks\(^1\). The ground of this literature talks of the social dimension of human beings, as agents that are constantly interacting with others in their daily life. This is the case in many social and economic situations, from an agent that gets information on job opportunities from friends and relatives, to a firm that chooses which technology to adopt depending on that used by its counterparts. However, the maintenance of these relationships is not always costless. Thus, for example, businessman are required to travel to contact and open new markets, and ordinary people are supposed to make phone calls and meet relatives and friends from time to time to maintain fluent relationships.

The literature on networks has devoted a lot of attention to address the issue of (costly) link formation in the context of coordination games. That is to say, several papers analyze link formation in situations in which, when any two players get linked, they play a 2x2 coordination game. Note that for the network to play a role here, it is required that each agent takes the same action in all her interactions, as otherwise, we would be dealing with independent games\(^2\). One of the main dimensions in which the existing papers in the literature differ is the way the cost sharing among agents is addressed. We remark three works on this respect.

Goyal and Vega-Redondo (2005) analyze a one-stage game where both, links and actions, are chosen simultaneously. They consider a one-sided link formation model, where links can be made on an individual basis, meaning that, whenever an agent decides to form a link, she unilaterally meets the entire cost of the link. They assume that interaction yields positive benefits, which means that no agent has incentives to refuse to form a link when she receives a link proposal. Their analysis yield three possible equilibrium outcomes (apart from the trivial outcome of the empty network, when the cost is too high): (i) the complete network with all the players coordinated on the efficient action\(^3\), (ii) the complete network with all the players coordinated on the risk-dominant action, and (iii) a network formed by two complete components\(^4\), one component of players choosing the efficient action

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(1) See, for instance, Jackson (2008) and Goyal (2007).
(2) This can be interpreted as the choice of a standard. For example, in the case of a population of researchers, it may represent the decision of whether to use PC or Mac. As it is clear from the example, once the choice is taken, the election applies to all the links the researcher has, meaning that she will use the hardware to work on all the papers co-authored with other researchers.
(3) The proposed 2x2 coordination game is symmetric and presents two equilibria: one efficient, the other risk-dominant.
(4) A complete component is a set of players such that, any two players in the set are linked and no agent in the set is linked to any agent outside the set.
and the other component of players choosing the risk-dominant action. To deal with the issue of equilibrium selection, they introduce a (perturbed) dynamics and find that when the link cost is high, the unique stochastically stable state is state (i); whereas if the link cost is low, state (ii) is selected.

Jackson and Watts (2002) consider a two-sided link formation model in which whenever a link forms, the two players involved in the link meet part (one half) of the cost. Therefore, in their case, the formation of a link requires the consent of the two players\(^5\). They directly analyze the model in a dynamic framework and find that, depending on parameters, either the dynamics selects a unique stochastically state (such a state being state (i) or state (ii) defined above), or both states coexist in the long run\(^6\).

Finally, Meléndez-Jiménez (2008) considers a model in which actions (for the coordination game) and link formation take place at different stages. It is assumed that, in the first stage, agents choose actions and, given the pattern of action, in the second stage, each pair of players go into a bargaining process in which, either they agree on how to share the cost of a link (therefore the link forms), or the link is not formed. The agents' shares of the link costs are therefore endogenous. As for the bargaining process, the author focuses on the symmetric Nash bargaining solution, which has two particularities. First, whenever a link is profitable (i.e., the aggregate payoff to the agents involved exceeds the cost), the link forms. Second, the agent who receives the higher (gross) payoff from the link meets a higher part of the cost. The results show that, under this specification, there are only two equilibrium outcomes: states (i) and (ii) defined above. The dynamic version of the model yields uniqueness of the stochastically stable states. In particular, when the cost is higher than a certain threshold (the risk-dominance premium of the coordination game), efficiency is achieved whereas, when it is lower, risk-dominance considerations prevail.

The present paper builds on the static version of the model of Meléndez-Jiménez (2008), to consider the generalized Nash bargaining solution. In particular, we allow the bargaining power of an agent to depend on whether she is the proposal of the link or she receives the link proposal. We consider a three-stage game with the following structure. In the first stage, each agent chooses a standard (an action for the coordination game). Given the choices on standards, in the second stage,

\(^5\) In Droste, Gilles and Johnson (2000), the same cost scheme is assumed but, in addition, this work introduces a spatial location of agents.

\(^6\) The reported results correspond to the case of constant costs of maintaining links, in analogy to the present paper. Nevertheless, they also analyse other cost schemes as, for example, the case where the total cost an agent meets in all her links is increasing and convex in the number of links she has.
each agent decides whom to propose a link (i.e., who she wants to interact with). In the third stage, for each proposed link, agents bargain over the sharing of the cost. Here, we assume that the bargaining power of an agent depends on whether she is the proposal of the link. In particular, we consider that the proposer of the link is in a weaker position. It allows us to make the link proposal decision of any agent contingent on the observed pattern of actions. As in the already mentioned works, the (gross) payoffs from the interaction between two agents are represented by a 2x2 symmetric coordination game in which we identify the actions with the chosen standards. This game is characterized by two pure strategy Nash equilibria: one efficient, the other risk-dominant.

Note that, in the present paper, when the bargaining power of the proposer is set to be equal to 1/2, the results are those in the static version of the model of Meléndez-Jiménez (2008). Additionally, when the bargaining power of the proposer is set to be 0, the present model represents a situation close to that analysed in Goyal and Vega-Redondo (2005). The difference being that, in the present paper, the proposer of the link does not necessarily meet the entire cost; whereas in their case, the proposer pays it entirely. Note that, in the present model, a link forms whenever the aggregate (gross) payoff to both agents exceeds the cost. Additionally, as we do not allow for side payments, the maximum share of the cost an agent can meet is her individual gross payoff (from the bilateral coordination game). Then, in the present work, when the payoff to the proposer is lower than the cost, even if her bargaining power is 0, the proposer only meets part of the cost.

Our results determine five classes of equilibrium outcomes (apart from the trivial equilibrium of the empty network that arises when the cost of the link exceeds the maximum aggregate gross payoff). The first two classes of equilibria correspond to the most intuitive equilibrium outcomes, which are also equilibria of former models of network formation in the context of coordination games. These equilibria are characterized by the emergence of the complete network and the fact that all the agents choose the same standard. We denote these classes of equilibria by type I outcome and type II outcome, depending on whether the chosen standard is the efficient or the risk-dominant action, respectively. The other three equilibria arise as a consequence of actions and link proposals being chosen at different stages of the game. We denote by type III outcome an equilibrium in which a network of two components forms: a complete component of agents choosing the efficient

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(7) Since Meléndez-Jiménez (2008) assumes the symmetric Nash bargaining solution, in his model it makes no difference who is the proposer of a given link and, therefore, the second and third stages of the game collapse.

(8) The proposer of the link has (equal or) lower bargaining power than the respondent.
action and a complete component of agents choosing the risk-dominant action.\footnote{Note that this equilibrium is also achieved in Goyal and Vega-Redondo (2005).} We denote by type IV outcome an equilibrium in which the complete network forms but agents choose different standards. Finally, we denote by type V outcome an equilibrium in which all the agents that choose the efficient action get linked to all the population, and all the agents that choose the risk-dominant action only get linked to the agents that choose the efficient one. Summarizing, the fact that link proposers have lower bargaining power than link receivers, allows to strategically support a richer variety of equilibrium outcomes, as compared to other models of network formation.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 provides an adaptation of the Nash bargaining solution to our model. Section 4 characterizes the equilibrium outcomes of the game. Finally, Section 5 concludes.

2. THE MODEL

Let \( N = \{1, \ldots, n\} \) be the set of players, where \( n \geq 2 \). We assume agents to be risk neutral. The model is a three-stage game. In the first stage, each player \( i \in N \) chooses an action \( a_i \) from the available action space \( A = \{\alpha, \beta\} \). Upon observing the pattern of actions \( a = (a_1, a_2, \ldots, a_n) \in A^n \), in the second stage, each player proposes links, i.e., each player \( i \in N \) chooses a vector \( g_i(a) = (g_{i,1}(a), g_{i,2}(a), \ldots, g_{i,n}(a)) \) where \( g_{i,j}(a) = 1 \) if, given \((a_i, a_j)\), player \( i \) proposes a link to player \( j \) and \( g_{i,j}(a) = 0 \) otherwise. We fix \( g_{i,i} = 0 \) \( \forall \ i \in N \). In the third stage, the cost sharing of each link is determined according to the generalized Nash bargaining solution, where the bargaining power of agent \( i \) in a link \( ij \) is:

\[
\begin{align*}
\gamma & \quad \text{if} \quad g_{ij} = 1 \quad \text{and} \quad g_{ji} = 0 \\
(1-\gamma) & \quad \text{if} \quad g_{ij} = 0 \quad \text{and} \quad g_{ji} = 1 \\
1/2 & \quad \text{if} \quad g_{ij} = g_{ji} = 1
\end{align*}
\]

We assume \( \gamma \leq 1/2 \) as, otherwise, in any equilibrium strategy for any possible link between a pair of agents \( i, j \in N \), either both agents decide to propose the link \( g_{ij} = g_{ji} = 1 \) and the symmetric Nash bargaining solution applies, or none of them proposes the link; which would yield the static model of Meléndez-Jiménez (2008). To characterize the conditions for a link to form, we first define the costs and benefits to the players taking part of the link. Given the actions chosen by any two players \( i \) and \( j \), the establishment of a link \( ij \) between them conveys:
(i) A gross payoff to the agents determined by the symmetric function \( \pi : A \times A \rightarrow \mathbb{R}^+ \). This function is represented by the following matrix:

\[
\begin{array}{c|cc}
\alpha & \beta \\
\hline
\alpha & d & e \\
\beta & f & b \\
\end{array}
\]

where the entries represent the payoffs to player \( i \), \( \pi(a_i,a_j) \), and \( j \), \( \pi(a_j,a_i) \), given actions \( a_i \in A \) and \( a_j \in A \). It is assumed that:

\[
d > f, b > e, d > b, b + f > d + e.
\]

Note that conditions \( d > f \) and \( b > e \) reflect the fact that \( \pi(\cdot) \) represents the payoff matrix of a 2x2 symmetric coordination game. Condition \( d > b \) means that \((\alpha, \alpha)\) is the Pareto efficient equilibrium of this game; whereas condition \( b + f > d + e \) says that \((\beta, \beta)\) is the risk dominant equilibrium, as defined by Harsanyi and Selten (1988).

(ii) A fixed cost \( c > 0 \) that has to be borne in some feasible way, \( c = c_{ij} + c_{ji} \), between the involved players. We denote by \( c_{ij} \) the part of the cost paid by player \( i \) in the link \( ij \), and by \( c_{ji} \) the part of the cost paid by player \( j \). We assume \( c_{ij} \geq 0 \) and \( c_{ji} \geq 0 \). We therefore exclude the possibility of agents subsidizing others in order to form a link.\(^10\)

As mentioned, we consider that cost sharing results from the generalized Nash bargaining solution, i.e., in the third stage of the game, we take an axiomatic approach. It allows us to make the analysis quite general, as we do not have to explicitly define a bargaining procedure. Moreover, note that the decision of agents in the second stage \( g(a) = (g_1(a), g_2(a), ..., g_n(a)) \) results in a network structure.

A strategy profile is summarized by a pattern of actions and a vector of link proposals for each agent given any possible pattern of actions, i.e., \( s = \{\hat{a}, \{g(a)\} \} a \in A^i \), where \( \hat{a} \in A \). Let \( S \) represent the space of strategy profiles. Given \( s \in S \), the total payoff \( \Pi_i(s) \) player \( i \in N \) gets in this model is the sum of the payoffs she gets from all her formed links, net of the shares of the costs she pays (according to the generalized Nash bargaining solution and her link proposals), i.e.\(^11\)

\[
\Pi_i(s) = \sum_{\{a\}} \max\{g_i(a), g_j(a)\} \cdot (\pi(a_j,a) - c_{ij}).
\]

\(^{10}\) Note that \( c_{ij} < 0 \) would convey that player \( i \) gets additional profits from the link \( ij \) which, as \( c_{ij} + c_{ji} = c \), would come from player \( j \).

\(^{11}\) Note that \( c_{ij} \) is not defined when the link \( ij \) is not formed. In order to precisely state expression (2), assume without loss of generality that in this case \( c_{ij} = 0 \). This does not have any influence on the results, since it will belong to a zero-component of the expression \( \max\{g_i, g_j\} = 0 \).
We solve the model using a backward induction argument. In Section 3 we solve the third stage of the game, i.e., given a pattern of actions and link proposals, we analyze the sharing of link costs induced by the generalized Nash bargaining solution. Section 4 considers the resulting cost sharing and obtains the subgame perfect equilibria of stages 1 and 2.

3. COST SHARING

Taken as given the pattern of actions and the link proposals, in this section we analyze how a pair of players shares the cost $c$ of a link. The cost sharing depends on two points: (i) The actions chosen by the involved agents and the function $\pi$, and (ii) who proposed the link.

We assume the generalized Nash bargaining solution. To apply this solution concept allows us to analyze a wide range of different bargaining powers, depending on who proposes the link. Additionally, the generalized Nash bargaining solution displays the feature that the resulting cost division depends on two aspects, who proposes the link and the gross payoff each agent can get. Thus, ceteris paribus, the higher the gross benefit an agent obtains from the link, the (weakly) higher the part of the cost she will bear; and the higher the bargaining power of an agent, the lower the cost she will meet. Note that the generalized Nash bargaining solution is defined to study the sharing of positive benefits among two players whereas, in our case, we are interested in the analysis of cost sharing. This means that we first have to adapt our model to the generalized Nash bargaining solution. To this aim, we define the feasible net payoff set, over which players bargain.

Consider a pair of agents, $i, j \in N$, that have chosen actions $a_i, a_j \in \{\alpha, \beta\}$, respectively. Without loss of generality, assume $g_{i,j}(a_i) = 1$ and $g_{j,i}(a_j) = 0$, i.e., agent $i$ is the proposer of the link. We denote by $\hat{\pi}(a_i, a_j, c_{ij})$ and $\hat{\pi}(a_j, a_i, c_{ji})$ the net benefit of the link to agents $i$ and $j$, respectively. Let

$$\hat{\pi}(a_i, a_j, c_{ij}) = \begin{cases} \pi(a_i, a_j) - c_{ij} & \text{if the link } ij \text{ is formed} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\pi}(a_j, a_i, c_{ji}) = \begin{cases} \pi(a_j, a_i) - c_{ji} & \text{if the link } ij \text{ is formed} \\ 0 & \text{otherwise} \end{cases}$$

where $\pi(a_i, a_j)$ and $\pi(a_j, a_i)$ represent the payoffs obtained by each player from the 2x2 symmetric coordination game (cf. Table 1). Note that a necessary condition for the link $ij$ to form is that, given the action chosen by each of the two players, the following inequality holds:
\[ \pi(a, a) + \pi(a, a) \geq c \]  

That is to say, when the aggregate gross payoff to both players from the link exceeds the cost it conveys,\(^{12}\) there exists some surplus over which players \(i\) and \(j\) can bargain. In this situation, they will have to agree on how to divide \(c\) into \(c_i^\prime\) and \(c_j^\prime\). Obviously, an agent would not accept a link if her share of the cost exceeds her associated gross benefit. Thus, if (3) is satisfied, in equilibrium, the link will be proposed and cost sharing will be such that \(\pi(a, a) \geq c_i^\prime\) and \(\pi(a, a) \geq c_j^\prime\).

We now define the bargaining problem \((X, d)\) between players \(i\) and \(j\). The disagreement value for each player is equal to 0, as this is the net payoff they obtain if the link is not formed, i.e., \(d=(0,0)\). The set of feasible net payoff pairs is

\[ X(a_i, a_j) = \left\{ \left( \hat{\pi}_i, \hat{\pi}_j \right) \in \Re^2_+ \mid \begin{align*} \hat{\pi}_i + \hat{\pi}_j &\leq \pi(a_i, a_j) + \pi(a_j, a_i) - c \\ \hat{\pi}_i &\leq \pi(a_i, a_i) \\ \hat{\pi}_j &\leq \pi(a_j, a_i) \end{align*} \right\} \]  

(4)

where, for notational convenience, we write \(\hat{\pi}_i = \pi(a_i, a_i, c)\) and \(\hat{\pi}_j = \pi(a_i, a_i, c)\). Note that side payments are not allowed. Then, the net payoff an agent gets from a link cannot exceed her gross payoff, i.e., \(\hat{\pi}_i \leq \pi(a_i, a_i)\) and \(\hat{\pi}_j \leq \pi(a_j, a_i)\).\(^{13}\) Figure 1 below depicts the set of feasible payoffs \(X(a_i, a_j)\), for the possible relations between \(\pi(a_i, a_j), \pi(a_j, a_j)\) and \(c\).\(^{14}\)

We now turn to calculate the values of \(c_i\) and \(c_j\) that result from the disagreement value \(d=(0,0)\) and the set \(X(a_i, a_j)\), as defined by (4). We first characterize the net payoffs, \(\hat{\pi}_i\) and \(\hat{\pi}_j\), players \(i\) and \(j\) obtain from the link \(ij\), if it forms. The generalized Nash bargaining solution is defined as:\(^{15}\)

\[ \hat{\pi}^{N} = \argmax_{\pi_i \in X_i, \pi_j \in X_j} \left( \pi_i - d \right)^\gamma \cdot \left( \pi_j - d \right)^{1-\gamma} = \argmax_{\pi_i \in X_i, \pi_j \in X_j} \hat{\pi}_i^{\gamma} \cdot \hat{\pi}_j^{1-\gamma} \]

Note that, when \(\pi(a_i, a_j) + \pi(a_j, a_j) \geq c\), it is always possible to find at least one distribution \(c_i + c_j = c\) such that \(\pi(a_i, a_j) - c_i \geq 0\) and \(\pi(a_j, a_j) - c_j \geq 0\). This implies that players will agree to form the link (as the disagreement value conveys zero-payoff to each player). Applying the generalized Nash bargaining solution to the bargaining problem \((X, d)\), with \(d=(0,0)\) and feasible set \(X\) defined by (4), we obtain \(\hat{\pi}_i\) and

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\(^{12}\) Note that expression (3) may be rewritten as \(\hat{\pi}(a, a, c) + \hat{\pi}(a, a, c) \geq 0\), since \(c_i + c_j = c\).

\(^{13}\) If we were to allow for transfers, the form of the set \(X(a, a)\) would always be as depicted in Figure 1(a), independently of whether values \(\pi(a, a)\) and \(\pi(a, a)\) were higher or lower than \(c\). Note also that these conditions are equivalent to \(\min\{c, c\} \geq 0\).

\(^{14}\) In the figure we write \(\pi = \pi(a, a), \pi = \pi(a, a)\) and \(X = X(a, a)\).

\(^{15}\) See, for example, Osborne and Rubinstein (1990).
\(\hat{x}_i\) as follows. Without loss of generality, consider \(\pi(a,a) \geq \pi(a,a)\). There are four possible situations: (Case 1) \(a_i=a_j=\alpha\), (Case 2) \(a_i=a_j=\beta\), (Case 3) \(a_i=\alpha, a_j=\beta\), and (Case 4) \(a_i=\beta, a_j=\alpha\). Additionally, note that the maximization problems is:

\[
\begin{align*}
\max H(\hat{x}_i, \hat{x}_j) &= \hat{x}_i \gamma \cdot \hat{x}_j \gamma \\
\text{s.t. } (\hat{x}_i, \hat{x}_j) &\in X(a,a); \hat{x}_i, \hat{x}_j \geq 0.
\end{align*}
\]

**FIGURE 1**
**FEASIBLE NET PAYOFF SET**

We obtain that the strictly Pareto-efficient points of \(X(a,a)\) are included in the line \(\hat{x}_i + \hat{x}_j = \pi(a,a) + \pi(a,a) - c\). Thus, \(\frac{d\hat{x}_i}{d\hat{x}_j} = 1\). As the generalized Nash bargaining solution must be a (strictly) Pareto-efficient point, we have two possibilities:

- If the solution is interior, it must satisfy:

\[
\begin{align*}
\frac{d\hat{x}_i}{d\hat{x}_j} &= -\frac{\frac{\partial H(\hat{x}_i, \hat{x}_j)}{\partial \hat{x}_i}}{\frac{\partial H(\hat{x}_i, \hat{x}_j)}{\partial \hat{x}_j}} \\
&= -\frac{\gamma \hat{x}_j}{(1-\gamma) \hat{x}_i} \\
&= \hat{x}_i = \frac{\gamma}{1-\gamma} \hat{x}_j,
\end{align*}
\]

and hence

\[
(\hat{x}_i^N, \hat{x}_j^N) = \left(\gamma \left(\pi(a_i, a_j) + \pi(a_j, a_i) - c\right), (1-\gamma) \left(\pi(a_i, a_j) + \pi(a_j, a_i) - c\right)\right)
\]
- If we have a corner solution, then:

$$\left( \hat{r}_{i}^{N}, \hat{r}_{j}^{N} \right) = \begin{cases} 
\left( \pi(a_{i}, a_{j}), \pi(a_{j}, a_{i}) - c \right) & \text{if } (1 - \gamma)(\pi(a_{i}, a_{j}) + \pi(a_{j}, a_{i}) - c) < \pi(a_{j}, a_{i}) - c \\
\left( \pi(a_{i}, a_{j}) - c, \pi(a_{j}, a_{i}) \right) & \text{if } \gamma(\pi(a_{i}, a_{j}) + \pi(a_{j}, a_{i}) - c) < \pi(a_{i}, a_{j}) - c 
\end{cases}$$

With this result at hand, we now characterize the four aforementioned cases (recall that $g_{i}(a)=1$ and $g_{j}(a)=0$).

**Case 1:** $a_{i}=a_{j}=\alpha$ (assuming $2d>c$)

$$\left( \hat{r}_{i}^{N}, \hat{r}_{j}^{N} \right) = \begin{cases} 
(d - c, d) & \text{if } [d > c] \land [\gamma < (d - c)/(2d - c)] \\
(\gamma(2d - c), (1 - \gamma)(2d - c)) & \text{otherwise}
\end{cases}$$

**Case 2:** $a_{i}=a_{j}=\beta$ (assuming $2b>c$)

$$\left( \hat{r}_{i}^{N}, \hat{r}_{j}^{N} \right) = \begin{cases} 
(b - c, b) & \text{if } [b > c] \land [\gamma > (b - c)/(2b - c)] \\
(\gamma(2b - c), (1 - \gamma)(2b - c)) & \text{otherwise}
\end{cases}$$

**Case 3:** $a_{i}=\alpha$, $a_{j}=\beta$ (assuming $e+f>c$)

$$\left( \hat{r}_{i}^{N}, \hat{r}_{j}^{N} \right) = \begin{cases} 
(e - c, f) & \text{if } [e > c] \land [\gamma < (e - c)/(e + f - c)] \\
(e, f - c) & \text{if } [f > c] \land [\gamma > e/(e + f - c)] \\
(\gamma(e + f - c), (1 - \gamma)(e + f - c)) & \text{otherwise}
\end{cases}$$

**Case 4:** $a_{i}=\beta$, $a_{j}=\alpha$ (assuming $e+f>c$)

$$\left( \hat{r}_{i}^{N}, \hat{r}_{j}^{N} \right) = \begin{cases} 
(f - c, e) & \text{if } [f > c] \land [\gamma < (f - c)/(e + f - c)] \\
(\gamma(e + f - c), (1 - \gamma)(e + f - c)) & \text{otherwise}
\end{cases}$$

Note that, once we have the net payoffs that a pair of agents obtain when they choose a pair of actions ($a_{i}, a_{j}$), we can uniquely define the share of the cost each agent meets: $c_{i} = \pi(a_{i}, a_{j}) - \hat{r}_{i}^{N}$ and $c_{j} = \pi(a_{i}, a_{j}) - \hat{r}_{j}^{N}$. Given these results, in the next section we analyse the equilibria of the game.

4. EQUILIBRIUM ANALYSIS

We now obtain the subgame perfect equilibria of the three stage game. We take as given the distribution of the link cost obtained in the previous section. The strategy of an agent is therefore conformed by a tuple $(\hat{a}_{i}, g(a)_{a \in A})$, i.e., an action $\hat{a}_{i} \in A$ for the first stage and a set of links $g(a)$ to propose for any possible pattern of actions $a \in A^{n}$, in the second stage. Next proposition shows a necessary condition for a strategy profile to be a subgame perfect equilibrium:
Proposition 1  Let \{\hat{a}, \{g(a)\}_{a \in A_i}\} be a subgame perfect equilibrium. For each \(i,j \in N\) and \(a \in A_i^a\),

(i) If \(\pi(a, a) + \pi(a, a) > c\), then \(g_{ij}(a) + g_{ji}(a) = 1\).

(ii) If \(\pi(a, a) + \pi(a, a) < c\), then \(g_{ij}(a) = g_{ji}(a) = 0\).

Proof: (i) \(\pi(a, a) + \pi(a, a) > c\) Let . Assume that the link is not proposed \((g_{ij}(a) = g_{ji}(a) = 0)\), then \(\hat{\pi}_i = \hat{\pi}_j = 0\). In contrast, if the link is proposed, both players obtain a non-negative net payoff. Then, a player would have incentives to deviate to propose the link. Now assume that both players propose the link \((g_{ij}(a) = g_{ji}(a) = 1)\). In this case, each player has a bargaining power of 1/2. However, a player has incentives to deviate to not propose the link, hence, being the respondent and enjoying a bargaining power of \((1-\gamma)\geq 1/2\). (ii) This point is trivial. ■

Corollary 1  If \(c > 2d\), a strategy profile \(s = \{\hat{a}, \{g(a)\}_{a \in A_i}\} \in S\) is an equilibrium if and only if \(g_{ij}(a) = 0\), for any \(i,j \in N\) and any \(a \in A_i^a\).

Proof: For any \((a_i, a_j) \in A_i^2\), \(\pi(a_i, a_j) + \pi(a_j, a_i) \leq 2d\). Then, if \(c > 2d\), condition (3) above, which is necessary for a link to form, is not satisfied. ■

Note that, in the present game, link proposals \(g(a)\) are contingent on chosen actions \(a\). This fact, together with the feature that the proposer of a link has a lower bargaining power than the responder \((\gamma \leq 1/2)\), yields a richer set of equilibrium outcomes, as compared to the static version of Meléndez-Jiménez (2008), who considers the case of symmetric bargaining power, and other related works.\(^{16}\) To this respect, it is worth noting that the class of bilateral coordination games played in networks usually present two possible equilibrium outcomes (provided that the cost is not so high that no link is formed): (i) all the agents choose the efficient action \(\alpha\) and the complete network arises (type I outcome) and (ii), all the agents choose the risk-dominant action \(\beta\) and the complete network arises (type II outcome). As we show below, these are also equilibria in our framework but, additionally, we obtain other classes of equilibria that allow the coexistence of different actions. We describe them next:

Type III outcome,\(^{17}\) which corresponds to the case in which there are a set of agents choosing \(\alpha\), \(N_\alpha \subset N\), and a set of agents choosing \(\beta\), \(N_\beta = N \setminus N_\alpha\), such that any \(i,j \in N_\alpha\) and any \(k,l \in N_\beta\) get linked, but no link between agents \(i\) and \(k\), with \(i \in N_\alpha\) and \(k \in N_\beta\), is formed. That is to say, there is a network with two components, a complete component of \(\alpha\)-players and a complete component of \(\beta\)-players.

---

\(^{16}\) See, for example, Goyal and Vega-Redondo (2005) or Jackson and Watts (2002).

\(^{17}\) This equilibrium outcome is also an equilibrium in Goyal and Vega-Redondo (2005).
Type IV outcome, which corresponds to the case in which the complete network arises, with a set of agents choosing \( \alpha \), \( N_\alpha \subset N \), and a set of agents choosing \( \beta \), \( N_\beta = \mathcal{M} \, N_\alpha \).

Type V outcome, which corresponds to the case in which there are a set of agents choosing \( \alpha \), \( N_\alpha \subset N \), and a set of agents choosing \( \beta \), \( N_\beta = \mathcal{M} \, N_\alpha \), and a connected but not complete network arises, where each \( \alpha \)-player gets connected to all the population, but there are no links between \( \beta \)-players.

4.1. Type I outcome

The first possible equilibrium outcome consists of all the agents choosing the efficient action and forming the complete network. This situation clearly yields the efficient outcome when \( 2d > c \), as, in this case, it represents the maximum achievable aggregate payoff. Next proposition characterizes the parameter range where this outcome can be sustained as an equilibrium.

Proposition 2 Assume \( 2d > c \). The complete network with all the agents coordinated on the efficient action can be sustained as equilibrium.

The proof is shown in the Appendix. Succinctly sketched, we define a set of strategy profiles \( S_1 \subset S \) characterized by the fact that all the agents choose \( \alpha \) and, for any pattern of actions, a link forms if and only if the aggregate payoff of both agents involved exceeds the cost (in the equilibrium path it is clear that, given \( 2d > c \), the complete network arises). Moreover, each link may be only proposed by one player.\(^\text{18}\) In order to achieve this outcome as an equilibrium for the widest parameter range we assume that, in any subgame where an agent deviates to \( \beta \), she becomes the proposer of all the (profitable links) she may form. We show that, even in the extreme case in which, in the equilibrium path, a player proposes links to all the population, there are no incentives to deviate.

4.2. Type II outcome

In this section we analyze under which circumstances a situation characterized by the complete network with all the agents coordinated in the risk-dominant action \( \beta \), can be strategically supported.

Proposition 3 Assume \( 2b > c \). The complete network with all the agents coordinated on the risk-dominant action can be sustained as equilibrium if and only if one of the following conditions (i)-(iii) holds:

\[
\begin{align*}
(i) \quad c &> e+f
\end{align*}
\]

\(^{\text{(18)}}\) Note that these conditions are necessary in order to attain subgame perfection.
The proof is shown in the Appendix. We now sketch the main arguments. First, we define a particular set of strategy profiles \( S_2 \subset S \), which provides a type II outcome. We prove that these strategies constitute an equilibrium if and only if one of the aforementioned conditions hold (sufficient condition). Finally, we prove that if a strategy profile \( s \in S \) providing the type II outcome is an equilibrium, then any strategy in the set \( S_2 \) is an equilibrium (necessary condition).

Note that for the complete network with all the agents coordinated on the risk-dominant action to be an equilibrium, it is necessary (and sufficient) that one of the following three conditions hold: (i) the cost exceeds the aggregate payoff of choosing different actions \( (c>e+f) \), as, in this case, a link between an \( \alpha \)-player and a \( \beta \)-player does not form (hence, no player has incentives to deviate); (ii) the bargaining power of the proposer is sufficiently low, such that no agent has incentives to deviate to \( \alpha \) (in which case, she would be the proposer of all her links); (iii) the cost is sufficiently low \( (c<2(b-e)) \), which assures that, even if an agent who deviates to \( \alpha \) pays no cost (because parameters are such that the respondent entirely meets the cost, i.e., \( f>c \) and \( \gamma>\left(\frac{2b-c}{2(e+f-c)}\right) \)), the benefits from deviating are smaller than those of no deviating.

Next corollary shows that, when the aggregate payoff of two agents that coordinate in the risk dominant action is higher than their aggregate payoff when choosing different actions, type II outcome can be sustained as equilibrium.

**Corollary 2** Assume \( 2b>c \). If \( 2b>e+f \), the complete network with all the agents coordinated on the risk-dominant action can be sustained as equilibrium.\(^{19}\)

**Proof:** \( 2b>e+f \) implies \( (2b-c)/(2(e+f-c))>1/2 \). Hence \( \gamma<(2b-c)/(2(e+f-c)) \), which, from Proposition 3, is a sufficient condition for an equilibrium. \( \blacksquare \)

### 4.3. Type III outcome

We now consider the case in which a network embedded by two complete components arises: a complete component of \( \alpha \)-players and a complete component of \( \beta \)-players. In other words, there is a set of agents choosing \( \alpha \), \( N_\alpha \subset N \), and a set of agents choosing \( \beta \), \( N_\beta = N \setminus N_\alpha \), such that any \( i,j \in N_\alpha \) and \( k,l \in N_\beta \) get linked.

\(^{19}\) Moreover, when \( 2b>e+f \), it is not necessary to assume a symmetric distribution of link proposals in the equilibrium path in order to get an equilibrium (as we require for strategies in the set \( S_2 \)). The proof is very similar to the proof of Proposition 2 and, hence, is omitted.
but no link between two players $i$ and $k$, with $i \in N_\alpha$ and $k \in N_\beta$, forms. Let us first present an introductory result:

**Remark 1.** There exists equilibria such that the resulting network consists of two connected components only if $2b > c > e+f$.

The result directly follows from Proposition 1. Note that $2b>c$ is a necessary condition for all the players choosing $\beta$ being linked across themselves. Additionally, $c>e+f$ is necessary for no link existing between a player choosing $\beta$ and a player choosing $\alpha$.

We next analyze the conditions under which a network with a complete component of $\alpha$-players and a complete component of $\beta$-players can be strategically supported. To simplify the analysis, in the rest of the paper we will restrict our attention to the case $c>d$, which allows us to uniquely define the net payoffs for any pair of actions (cf. cases 1-4 in Section 3).\(^{(20)}\) Additionally, note that, in the one-sided link formation model proposed by Goyal and Vega-Redondo (2005), if $c>d$ the empty network arises. Hence, the presence of interaction in this parameter region is specific to our model. The next proposition characterizes the conditions under which this class of outcomes can be strategically supported. Let $n_\alpha = |N_\alpha|$

**Proposition 4** Assume $2b>c>e+f$ and $c>d$. A network formed by two complete components ($N_\alpha$ and $N_\beta$) can be supported as equilibrium if and only if:\(^{(21)}\)

$$\left\lfloor \frac{(1/2)(2b-c)}{\gamma(2d-c)+(1/2)(2b-c)} \right\rfloor \frac{n_\alpha}{n} > \left\lceil \frac{\gamma(2b-c)}{(1/2)(2d-c)+\gamma(2b-c)} \right\rceil$$

The proof is shown in the Appendix. The main arguments in the proof are the following. We first define a particular set of strategy profiles $S_3 \subseteq S$, which yields a type III outcome. We prove that, given $2b>c>\max\{e+f,d\}$, these strategies constitute an equilibrium if and only if the aforementioned (sufficient) condition holds. Finally, we prove that if a strategy profile $s \in S$ yielding a type III outcome is an equilibrium, then any strategy in set $S_3$ is an equilibrium (necessary condition).

We comment on this result. Note that the proportion of players who choose $\alpha$, $n_\alpha/n$, needs to be high enough so that no agent in the set $N_\alpha$ has incentives to deviate to $\beta$, i.e., $n_\alpha/n > \left\lfloor \gamma(2b-c)/(1/2)(2d-c)+\gamma(2b-c) \right\rfloor$. Additionally, this proportion has to be smaller than a threshold, $n_\alpha/n < \left\lceil (1/2)(2b-c)/(\gamma(2d-c)+(1/2)(2b-d)) \right\rceil$, such that no agent in the set $N_\beta$ has incentives to deviate to $\alpha$. Note that, when $\gamma=1/2$,

\(^{(20)}\) See the proof of Proposition 4 for an specification of these net payoffs
\(^{(21)}\) $[\cdot]$ and $\lceil \cdot \rceil$ represent the floor and ceiling functions.
both conditions cannot be simultaneously satisfied. Hence, in Meléndez-Jiménez (2008), this class of outcome is not sustained in equilibrium.

The next result determines the parameter region where the only equilibria are type I, II and III outcomes.

**Remark 2** If $2d>c+e+f$, then only type I, type II and type III outcomes as described above can be supported as equilibrium.

Note that $c+e+f$ implies that, in equilibrium, no link between agents choosing different actions forms. If $2d>c>2b$, the only equilibrium is of type I outcome, in which all the agents choose the efficient action and the complete network arises. Additionally, if $2b>c+e+f$, either all the agents choose the same action and the complete network arises (type I and II outcomes); or a network formed by two complete components composed of agents choosing different actions arises (type III outcome).

### 4.4. Type IV outcome

This section considers the outcome in which the complete network forms, with a set of agents choosing $\alpha$, $\alpha \subset N$, a set of players choosing $\beta$, $\beta \subset N \setminus \alpha$. We first present an introductory result:

**Remark 3.** There exist equilibrium outcomes such that the complete network arises and there are agents choosing different actions only if $\min[2b,e+f] > c$.

Note that this result follows directly from Proposition 1.

Restricting our attention to the case $c>d$ we now characterize the conditions under which this class of outcome can be sustained in equilibrium. To this aim, we differentiate two cases, depending on whether the aggregate gross payoff associated to two agents choosing $\beta$ exceeds the aggregate gross payoff associated to the case in which one agent chooses $\alpha$ and the other takes $\beta$ (i.e., $2b+e+f$). Propositions 5 and 6 formalize these results.

**Proposition 5.** Assume $c>d$ and $2b+e+f>c$. The complete network with coexistence of different actions can be sustained as equilibrium if one of the following conditions (i)-(ii) holds.

\[
\begin{align*}
(i) & \quad y < \frac{e+f-c}{(2d-c)+(e+f-c)} \quad \text{and} \quad \frac{n\alpha}{n} > \left[\frac{y(2b-(e+f))}{y(2(b-e-f)+c)+d-(c/2)}\right] \\
(ii) & \quad y > \frac{e+f-c}{(2d-c)+(e+f-c)} \quad \text{and} \quad \left[\frac{1/2(2b-c)-y(e+f-c)}{(1/2)(2b-c)+y(2d-c)-(e+f-c)}\right] > \frac{n\beta}{n} > \left[\frac{y(2b-(e+f))}{y(2(b-e-f)+c)+d-(c/2)}\right]
\end{align*}
\]

(22) This allows us to uniquely define net payoffs for any pair of actions (see proof of Proposition 5 in the Appendix)
The proof is shown in the Appendix. In the proof, we define a particular set of strategy profiles $S_4 \subset S$, which yield a type IV outcome. We prove that, given $2b > e+f > c > d$, these strategies constitute an equilibrium if and only if one of the aforementioned conditions holds (sufficient condition). In this case, we do not prove it is also a necessary condition.

We now comment on this result. In this case, the condition for an $\alpha$-player not finding it profitable to deviate from $s \in S$, requires a sufficient number of agents choosing $\alpha$, i.e., $n/\alpha > min \left\{ \gamma (2b - (e + f))/\gamma (2(b - e - f) + e + f), b > d \right\}$. On the other hand, the incentives of a $\beta$-player to deviate directly depend on $\gamma$. In particular, when the proposer has low bargaining power $\gamma < (e + f - c)/(2d - c + e + f - c)$, a $\beta$-player does not want to deviate. However, if $\gamma$ is high, a $\beta$-player would want to deviate unless there is a sufficient number of players choosing $\beta$, i.e., $n/\beta < min \left\{ 1/[2b - c - \gamma (e + f - c)], 2b - c \right\}$. Proposition 6 Assume $c > d$ and $e + f > 2b > c$. The complete network with coexistence of different actions can be sustained as equilibrium if and only if one of the following conditions (i)-(iv) holds.

(i) $\gamma < \min \left\{ \frac{2b - c}{2(e + f - c)} , \frac{e + f - c}{2(2d - c)} \right\}$

(ii) $\min \left\{ \frac{2b - c}{2(e + f - c)} , \frac{e + f - c}{2(2d - c + e + f - c)} \right\} > \gamma > \frac{2(e + f - b) - c}{2(2d - c)}$

(iii) $\frac{2b - c}{2(e + f - c)} \gamma > \max \left\{ \frac{2(e + f - b) - c}{2(2d - c - (e + f - c))} , \frac{e + f - c}{2(2d - c + (e + f - c))} \right\}$ and $\frac{n(2d - c - (e + f - c))}{n - 1} \gamma (e + f - c) > \frac{(1/2)(2b - c) - \gamma (e + f - c)}{(1/2)(2b - c) + \gamma (2d - c - (e + f - c))}$

(iv) $\frac{e + f - c}{(2d - c + (e + f - c))} \gamma > \max \left\{ \frac{2b - c}{2(e + f - c)} , \frac{2(e + f - b) - c}{2(2d - c)} \right\}$ and $\frac{n(2d - c - (e + f - c))}{n - 1} \gamma (e + f - c) > \frac{(1/2)(2b - c) - \gamma (e + f - c)}{(1/2)(2b - c) + \gamma (2d - c - (e + f - c))}$

The proof is in the Appendix. As previously, in the proof we define the set of strategy profiles $S_4 \subset S$ and show that, if $e + f > 2b > c > d$, any strategy in this set is an equilibrium if and only if one of the aforementioned conditions (i)-(iv) holds (sufficient condition). In this case, when $e + f > 2b$, any strategy $s \in S_4$ allows the type IV outcome to be sustained as an equilibrium for the widest parameter range. Hence, if any strategy providing a type IV outcome is an equilibrium, then any strategy in the set $S_4$ also constitutes an equilibrium (necessary condition).

We comment on this result. Note that no $\alpha$-player has incentives to deviate, even if in the equilibrium path she proposes links to all $\beta$-players. Expressions (i)-(iv) above characterize the conditions such that a $\beta$-player does neither want to deviate.
4.5. Type V outcome

In this case, we analyze a set of strategy profiles $S_β \subseteq S$ such that there is a set of agents choosing $α$, $N_α \subseteq N_β$, a set of agents choosing $β$, $N_β \cap N_α$, and the resulting network is connected but not complete. In this case, each $α$-player gets connected to all the population, and no link between $β$-players form. We first present an introductory result:

**Remark 4.** There exist equilibria such that the resulting network is connected but not complete only if $e+f>c>2b$. In this equilibrium, all players choosing $α$ get linked to all the population and there are no links between $β$-players.

Again, this result directly follows from Proposition 1.

Assuming $c>d$, we now characterize the conditions for this class of strategies ($S_β$) to constitute an equilibria.

**Proposition 7.** Assume $c>d$ and $e+f>c>2b$. A network consisting in a connected but not complete component can be sustained as equilibrium if and only if

$$
γ < \frac{e+f-c}{(2d-c)+(e+f-c)} \text{ and } \frac{n_α}{n-1} > \frac{γ(e+f-c)}{(e+f-c)−γ(2d-c)}
$$

The proof is shown in the Appendix. The procedure is analogous to the previous cases. We first define a particular set of strategy profiles $S_β \subseteq S$, that yield a type V outcome. Given $e+f>c=\max\{2b,d\}$, we next show that these strategies constitute an equilibrium if and only if the aforementioned conditions hold (sufficient condition). Last, we show that if a strategy profile $s \in S$ yielding a type V outcome is an equilibrium, then any strategy in the set $S_β$ is an equilibrium (necessary condition).

We comment on this result. In this case, no player in the set $N_β$ has an incentive to deviate. However, we do need conditions that prevent deviations of $β$-players. This condition requires a sufficient number of $α$-players, i.e. $n_α > (γ(e+f-c)/(e+f-c)−γ(2d-c))(n-1)$. Additionally, it is required the bargaining power of the proposer to be low enough.

We now note that the only equilibria of the game are the outcomes we analyze.

---

(23) Recall this assumption allows us to uniquely define net payoffs for any pair of actions. See the proof of Proposition 7 for an specification of such payoffs.
(24) Where each $α$-player gets connected to all the population, and no link between $β$-players form.
(25) Note that condition $γ<(e+f-c)/(2d-c)+(e+f-c)$ results in $(γ(e+f-c)/(e+f-c)−γ(2d-c))<1$, allowing the condition on $n_α$ to be feasible.
**Remark 5.** Assume $2d>c$. The only equilibria of the game are type I, II, III, IV and V outcomes.

The result follows from Proposition 1. Finally, we specify the parameters region in which incomplete networks can be sustained in equilibrium.

**Remark 6.** Only if $\max\{2b,e+f\} > c > \min\{2b,e+f\}$ there exist equilibria such that a network which is neither complete nor empty arises.

Note that if $2b>c>e+f$, there may be an equilibrium of the type III outcome, where the network consists of two complete components. Additionally, if $e+f>c>2b$, there may be an equilibrium of the type V outcome, where a connected but not complete network arises. However, if $c>\max\{2b,e+f\}$, the only links are those between $\alpha$-players. Hence, in this case, either the empty network arises (if all the players choose $\beta$ and/or $c>2d$) or the complete network arises (when $2d>c$, in which case all the players choose $\alpha$). Last, if $\min\{2b,e+f\}>c$, the complete network forms in equilibrium.

**5. CONCLUSIONS**

We consider a game in which players first choose an action for a coordination game and then propose costly links. We assume that players bargain over the distribution of the link cost, applying to this aim the generalized Nash bargaining solution. We consider that the bargaining power may differ across agents, depending on whether the player is the link proposer. Here we consider the whole range of bargaining power, from the case in which the link proposer has no bargaining power to the case in which all the agents have equal bargaining power. We characterize the set of subgame perfect equilibrium of the game. The fact that link proposals are contingent on the observed pattern of actions and the feature that the bargaining power may differ, depending on who proposes the link, yields a richer set of equilibrium as compared to other studies of network formation in the context of coordination games. Specifically, we obtain that there are equilibria where different actions coexist, either in separate components of the network, in the complete network, or in a connected but not complete network.

**REFERENCES**

APPENDIX

Proof of Proposition 2

To prove the proposition, we define a particular set of strategy profiles $S, \subseteq S$ which provide the type I outcome. Then we prove that if $2d>c$, any strategy $s \in S_i$ constitutes an equilibrium.

Let us first define the set of strategy profiles $S, \subseteq S$, such that $s=\{\hat{a}, \{g(a)\}_{a \in A^i}\} \in S_i$ if and only if the following conditions (I.i)-(I.iii) hold:

(I.i) $\hat{a}=a, \forall i \in N$, i.e. $\hat{a}=(a,a,..,a)$

(I.ii) $g_i(\hat{a})+g_{\hat{a}}(\hat{a}) = \begin{cases} 1 & \text{if } \pi(a_i, a_i) + \pi(a_j, a_i) \geq c \quad \forall \{i,j\} \subseteq N \text{ and } \forall \in A^i \\ 0 & \text{otherwise} \end{cases}$

(I.iii) $g_i(\hat{a}=\beta, a=a) = \begin{cases} 1 & \text{if } e+f \geq c \quad \forall \{i,j\} \subseteq N \\ 0 & \text{otherwise} \end{cases}$

That is to say, a strategy belongs to $S_i$ if all players choose the efficient action (I.i), and whenever the aggregate gross payoff from a link $ij$ exceeds the cost, the link is proposed by one and only one of the players involved (I.ii). Moreover we impose that, any player who deviates to chooses action $\beta$ becomes the proposer of any possible profitable link she may form (I.iii).

Assume $2d>c$. We claim that any strategy $s \in S_i$ constitutes an equilibrium. Clearly, the agent who would have higher incentives to deviate from a strategy profile $s=\{\hat{a}, \{g(a)\}_{a \in A^i}\} \in S_i$ is the agent who, in the equilibrium path, is proposing a higher number of links. To see this note that depending on the parameter conditions (either situation (S.i) or (S.ii) below, -cf. case 1 in Section 3-), the payoff an agent $i \in N$ obtains from $s$ is:

(S.i) If either \([2d-c>d]\) or \([d>c \text{ and } \gamma>(d-c)/(2d-c)]\):

\[
\Pi(s) = (\sum_{j=1}^{n} g_j(\hat{a})) \gamma (2d-c) + (n - (\sum_{j=1}^{n} g_j(\hat{a}))) (1-\gamma)(d-c)
\]

(S.ii) If \([d>c \text{ and } \gamma<(d-c)/(2d-c)]\)

\[
\Pi(s) = (\sum_{j=1}^{n} g_j(\hat{a})) (d-c) + (n - (\sum_{j=1}^{n} g_j(\hat{a}))) d
\]

And the payoff obtained if she deviates to choose \(\beta\) (note that in this case she becomes the proposer of all her links), depending on the parameters (either situation (S.iii), (S.iv) or (S.v) below, -cf. case 4 in Section 3- is:

(S.iii) If \([e+f>c]\) or \([f>c \text{ and } \gamma>(f-c)/(e+f-c)]\)

\[
\Pi_i^d = (n-1) \gamma (e+f-c)
\]

(S.iv) If \([f>c \text{ and } \gamma<(f-c)/(e+f-c)]\)

\[
\Pi_i^d = (n-1)(f-c)
\]

(S.v) If \(c>e+f\)

\[
\Pi_i^d = 0
\]

Note that the payoff of deviating is the same for all the population, but the payoff from following the strategy is increasing in the number of proposed links for the equilibrium pattern of actions, \(\sum_{j=1}^{n} g_j(\hat{a})\). Hence, let us consider a player \(i\) who is proposing links to all the population \(\sum_{j=1}^{n} g_j(\hat{a}) = n-1\) to show that this player does not have incentives to deviate to action \(\beta\). Clearly, if this is the case, the strategy \(s \in S_i\) is an equilibrium for any possible distribution of link proposals. For player \(i\) we rewrite the payoffs in the parameter ranges (S.i) and (S.ii):

(S.i) \(\Pi_i(s) = (n-1)\gamma(2d-c)\); \hspace{1cm} (S.ii) \(\Pi_i(s) = (n-1)(d-c)\)

Note that, if we compare the possible payoffs, for the different parameter ranges, from following the strategy and deviating we have:

If we consider that both (S.i) and (S.iii) hold:

\[
\Pi_i(s) = (n-1)\gamma(2d-c) > (n-1)\gamma(e+f-c) = \Pi_i^d
\]

If we consider that both (S.i) and (S.iv) hold:

\[
\Pi_i(s) > \Pi_i^d \text{ if and only if } \gamma>f-c)/(2d-c), \text{ which in this parameter range is always satisfied. Note that the combination of parameters in (S.i) and (S.iv) is possible only if } f>c \text{ and } (f-c)/(e+f-c)\gamma>(d-c)/(2d-c). \text{ Clearly, } (d-c)/(2d-c)>(f-c)/(2d-c).
\]
If we consider that both (S.ii) and (S.iii) hold:

\[ \Pi(S) > \Pi_i^d \] if and only if \( \gamma < (d-c)/(e+f-c) \), which in this parameter range is always satisfied. Note that if (S.ii) holds \( \gamma < (d-c)/(2d-c) \). Clearly \( (d-c)/(2d-c) < (d-c)/(e+f-c) \).

If we consider that both (S.ii) and (S.iv) hold:

\[ \Pi(S) = (n-1)(d-c)/(n-1)(f-c) = \Pi_i^d \]

Finally, if we consider that (S.v) holds, clearly \( \Pi(S) > \Pi_i^d = 0 \).

Hence, whatever the parameters of the model are, player \( i \) does not have incentives to deviate to choose action \( \beta \). As any other player’s incentives to deviate are lower than player \( i \)’s, the proposition follows. ■

**Proof of Proposition 3**

The proof has two steps. In the first step we define a particular set of strategy profiles \( S_2 \subseteq S \), which provide the type II outcome. We then show that an strategy \( s \in S_2 \) is an equilibrium if and only if one of the conditions (i)-(iii) in the proposition holds. Hence, we prove that this represents a sufficient condition. In the second step we show that if a strategy \( s' \in S \) resulting in the type II outcome is an equilibrium, then any strategy in the set \( S_2 \) constitutes an equilibrium. Thus, we prove that the proposition also provides necessary conditions.

**First step: sufficient condition**

Let us define the set of strategy profiles \( S_2 \subseteq S \) such that \( s=\{\hat{a}, \{g(a)\}_{a \in A^e}\} \in S_2 \) if and only if the following conditions (II.i)-(II.iv) hold:

**(II.i)** \( \hat{a} = \beta, \forall i \in N \), i.e. \( \hat{a} = (\beta, \ldots, \beta) \)

**(II.ii)** \( g_{ij}(\hat{a}) + g_{ji}(\hat{a}) = \begin{cases} 1 & \text{if } \pi(a_i,a_j) + \pi(a_j,a_i) \geq c \\ 0 & \text{otherwise} \end{cases} \quad \forall (i,j) \in N \) and \( \forall a \in A^e \)

**(II.iii)** \( g_{ij}(a_i = \alpha, a_j = \beta) = \begin{cases} 1 & \text{if } e + f \geq c \\ 0 & \text{otherwise} \end{cases} \quad \forall (i,j) \in N \)

**(II.iv)** \( \sum_{k \in N \setminus \{i\}} g_{ik}(\hat{a}) = (n-1)/2 \quad \forall i \in N \)

In words, a strategy belongs to \( S_2 \) if all players choose the risk-dominant action (II.i), and whenever the aggregate gross payoff from a link \( ij \) exceeds the cost, the link is proposed by one and only one of the players involved (II.ii). Moreover we impose that, any player who deviates to chooses action \( \alpha \) becomes the proposer of any possible profitable link she may form (II.iii). Additionally in any strategy \( s \in S_2 \),
in the equilibrium path, each player proposes half of the links she finally forms (II.iv). These two conditions are imposed in order to construct the strategy profiles which support the outcome type II for the widest parameter range.

Assume $2b>c$. A strategy $s=\{\tilde{s}, \{g(a)\}_{a \in A}\} \in S_2$ prescribes that each player proposes links to half of the population, i.e. $\sum_{a \in (1/2) N} g_a(i) = (n-1)/2 \forall i \in N$. Hence, as all players are choosing the same action ($b$), and the complete network forms, the payoff attained by every player is the same. Moreover the payoff obtained by any player who deviates to choose action $\alpha$ is also the same because, given $s$, the deviator becomes the proposer in any (profitable) link she may form. Hence consider an arbitrary player $i \in N$. We now look for the conditions such that this player does not want to deviate.

Depending on the parameter range (either situation (S.i) or (S.ii) below, -cf. case 2 in Section 3-), the payoff an agent $i \in N$ obtains from $s=\{\tilde{s}, \{g(a)\}_{a \in A}\}$ is:

(S.i) If either $[c>b]$ or $[b-c \text{ and } c>(2b-c)]$:

$$\Pi_i(s) = ((n-1)/2) \gamma (2d-c) + ((n-1)/2) (1-\gamma) (2d-c) = (n-1) (1/2) (2d-c)$$

(S.ii) If $[b-c \text{ and } c<(2b-c)]$

$$\Pi_i(s) = ((n-1)/2)(d-c) + ((n-1)/2)d = (n-1)(1/2)(2d-c).$$

Hence, for any parameter configuration agent $i$'s payoff is the same $\Pi_i(s) = (n-1)(1/2)(2d-c)$.

And the payoff obtained if player $i$ deviates to choose $\alpha$ (note that in this case he becomes the proposer of all her links), depending on the parameters (either situation (S.iii), (S.iv), (S.v) or (S.vi) below, -cf. case 3 in Section 3-) is:

(S.iii) If $[e+f>c]$ or $[b-c \text{ and } e \geq (e+f-c)]$ or $[e>c \text{ and } \gamma \in ((e-c)/(e+f-c), (e+f-c))]$

$$\Pi_i^2 = (n-1) \gamma (e+f-c)$$

(S.iv) If $[b-c \text{ and } e \geq (e+f-c)]$

$$\Pi_i^2 = (n-1)e$$

(S.v) If $[e>c \text{ and } \gamma \in ((e-c)/(e+f-c), (e+f-c))$

$$\Pi_i^2 = (n-1)(e-c)$$

(S.vi) If $[c>e+f]$

$$\Pi_i^2 = 0$$
We now look for the equilibrium conditions in each situation:

In (S.iii)  \( \Pi_i(s) > \Pi_i^d \) if and only if  \( \gamma < (2b-c)/(2(e+f-c)) \)

In (S.iv)  \( \Pi_i(s) > \Pi_i^d \) if and only if  \( 2(b-e) > c \)

In (S.v)  \( \Pi_i(s) = (n-1)(1/2)(2d-c) > (n-1)(e-c) = \Pi_i^d \)

In (S.vi)  \( \Pi_i(s) = (n-1)(1/2)(2d-c) > 0 = \Pi_i^d \)

Hence the first conclusion is that, when  \( c > e+f \),  \( s \) constitutes an equilibrium.

Now we claim that, assuming  \( 2b > c \), when  \( \gamma < (2b-c)/(2(e+f-c)) \),  \( s \) constitutes an equilibrium for any values of  \( \{d,b,e,f,c\} \). To see this, note that we just have to check for ranges of parameters defined in (S.iv). But note that, in (S.iv),  \( \gamma > e/(e+f-c) \). We know that in this situation an equilibrium requires  \( 2(b-e) > c \). So, assume this condition does not hold, i.e.  \( 2(b-e) < c \). Note that this implies that  \( e/(e+f-c) > (2b-c)/(2(e+f-c)) \), and hence if  \( \gamma < (2b-c)/(2(e+f-c)) \) and  \( 2(b-e) < c \) situation (S.iv) is not possible. Therefore  \( \gamma < (2b-c)/(2(e+f-c)) \) is a sufficient condition for  \( s \) to be an equilibrium.

Now we claim that, assuming  \( 2b > c \), when  \( 2(b-e) > c \),  \( s \) constitutes an equilibrium for any values of  \( \{d,b,e,f,c\} \). To see this note that we just have to check that this holds in ranges of parameters defined in (S.iii). But note that, in (S.iii),  \( \gamma < e/(e+f-c) \). We know that in this situation an equilibrium requires  \( \gamma < (2b-c)/(2(e+f-c)) \). So, assume this condition does not hold, i.e.  \( \gamma > (2b-c)/(2(e+f-c)) \). Note that  \( 2(b-e) > c \) implies that  \( e/(e+f-c) < (2b-c)/(2(e+f-c)) \), and hence if  \( 2(b-e) > c \) and  \( \gamma > (2b-c)/(2(e+f-c)) \) situation (S.iii) is not possible. Therefore  \( 2(b-e) > c \) is a sufficient condition for  \( s \) to be an equilibrium.

Now we claim that the fact that one of the conditions -(i), (ii) or (iii) in the proposition- holds is necessary for a strategy  \( s \in S_2 \) to constitute an equilibrium. Assume  \( e+f < c \) and  \( 2(b-e) < c \) and  \( \gamma > (2b-c)/(2(e+f-c)) \). Note that  \( (2b-c)/(2(e+f-c)) > (e-c)/(e+f-c) \), which means that in this case parameters do not allow for situation (S.v) (recall we are assuming  \( \gamma > (2b-c)/(2(e+f-c)) \)). The assumption  \( e+f < c \) implies situation (S.vi) is not possible. Hence given our assumptions, necessarily we have either situation (S.iii) or situation (S.iv). Then, as  \( 2(b-e) < c \) and  \( \gamma > (2b-c)/(2(e+f-c)) \), we already know that the strategy is not an equilibrium. This completes the first step of the proof.

**Second step: necessary condition**

We claim that if a strategy  \( s' \in S \) which provides a type II outcome is an equilibrium, then any strategy  \( s \in S_2 \) is an equilibrium. To see this note that conditions (II.i) and (II.ii) are necessary to obtain this outcome and to have subgame perfection,
respectively. Hence, strategy $s'$ must satisfy (II.i) and (II.ii). Note that condition (II.iii) makes any agent who deviates to obtain the worst possible outcome (since they become proposers in all their links). Thus, assume condition (II.iii) is also satisfied by $s'$. Clearly the payoff attained by any agent $i \in N$ who deviates to choose action $\alpha$ is the same (because any agent who deviates becomes the proposer in all her links). Differently the payoff an agent obtains from following the strategy is decreasing in the number of links she proposes. Hence the critical player is that one who is proposing a higher number of links. Any strategy in $S_2$ satisfies (II.iv), i.e. $\sum_{k \in N \setminus \{i\}} g_{ka}(\tilde{a}) = (n-1)/2 \ \forall \ i \in N$, which means that each player is proposing half of the links that she finally forms. Hence the incentives to deviate are the same across all players and weakly lower than the incentives of the critical player given $s$. This completes the proof. ■

Proof of Proposition 4

The proof has two steps. In the first step we define a particular set of strategy profiles $S_3 \subset S$, which provide the outcome type III. We then show that, if $2b>c>\max\{e+f,d\}$, any strategy $s \in S_3$ is an equilibrium if and only if one of the conditions (i)–(iii) in the proposition holds. Hence, we prove that this represents a sufficient condition. In the second step we show that if a strategy $s \in S$ resulting in outcome type III is an equilibrium, then any strategy in the set $S_3$ constitutes an equilibrium. Thus, we prove that the proposition also provides necessary conditions.

First step: sufficient condition

Let us define the set of strategy profiles $S_3 \subset S$ such that $s=\{\tilde{a}, \{g(a)\}_{a \in A}\} \in S_3$ if and only if the following conditions (III.i)–(III.vi) hold:

(III.i) \[ \tilde{a}_i=\alpha, \ \forall \ i \in N_{a} \subset N \]

(III.ii) \[ \tilde{a}_j=\beta, \ \forall \ j \in N_{\beta}=N\setminus N_{a} \]

(III.iii) \[ g_{ia}(\tilde{a})+g_{ja}(\tilde{a}) = \begin{cases} 1 & \text{if } a_i = a_j \in A, \\ 0 & \text{otherwise} \end{cases} \quad \forall \ i,j \in N \text{ and } \forall \ a \in A \]

(III.iv) \[ g_{ia}(\tilde{a}=\alpha, \tilde{a}=\alpha) = 1 \text{ if } i \in N_{a} \text{ and } j \in N_{a} \]

(III.v) \[ g_{ia}(\tilde{a}=\beta, \tilde{a}=\beta) = 1 \text{ if } i \in N_{\beta} \text{ and } j \in N_{\beta} \]

(III.vi) \[ \sum_{k \in N \setminus \{i\}} g_{ka}(\tilde{a}) = (n_{\alpha}-1)/2 \ \forall \ i \in N_{\alpha} \text{ and } \sum_{l \in N \setminus \{j\}} g_{la}(\tilde{a}) = (n_{\beta}-1)/2 \ \forall \ j \in N_{\beta} \]

where $n_{\alpha}=|N_{\alpha}|$ and $n_{\beta}=|N_{\beta}|=n-n_{\alpha}$. In words, a strategy belongs to $S_3$ if there is a set of players $N_{\alpha}$ who choose the efficient action and a set $N_{\beta}=N\setminus N_{\alpha}$ of players

\begin{align*}
\sum_{k \in N \setminus \{i\}} g_{ka}(\tilde{a}) & = (n_{\alpha}-1)/2 \ \forall \ i \in N_{\alpha} \\
\sum_{l \in N \setminus \{j\}} g_{la}(\tilde{a}) & = (n_{\beta}-1)/2 \ \forall \ j \in N_{\beta}
\end{align*}
who choose the risk-dominant action (III.i)-(III.ii), and, whenever the aggregate gross payoff from a link exceeds the cost, the link is proposed by one and only one of the players involved (III.iii). Moreover we impose that, any player who deviates to choose a different action becomes the proposer of any possible profitable link she may form (III.iv)-(III.v). Additionally in any strategy \( s \in S_j \), in the equilibrium path, each player proposes half of the links she finally forms (III.vi). These two conditions are imposed in order to attain the strategy profiles which support the outcome type III for the widest parameter range.

Assume \( 2b>c>e+f \) and \( c>d \). Let player \( i \) be the proposer of the link \( ij \). Then, from section 3 we have the following net payoffs:

- If \( (a,a) = (\alpha,\alpha) \) then \( (\hat{\alpha}_{i}, \hat{\alpha}_{j}) = (\gamma(2d-c),(1-\gamma)(2d-c)) \)

- If \( (a,a) = (\beta,\beta) \) then \( (\hat{\alpha}_{i}, \hat{\alpha}_{j}) = (\gamma(2b-c),(1-\gamma)(2b-c)) \)

- If \( (a,a) = (\alpha,\beta) \) then \( (\hat{\alpha}_{i}, \hat{\alpha}_{j}) = (0,0) \)

- If \( (a,a) = (\beta,\alpha) \) then \( (\hat{\alpha}_{i}, \hat{\alpha}_{j}) = (0,0) \)

First, we look for the condition such that an agent \( k \in N_{\alpha} \) does not want to deviate.

The payoff if agent \( i \) follows the strategy \( s \in S_j \) is:

\[
\Pi_i(s) = ((n_{\alpha}-1)/2)\gamma(2d-c)+((n_{\alpha}-1)/2)(1-\gamma)(2d-c) = (n_{\alpha}-1)(1/2)(2d-c)
\]

The payoff obtained if she deviates is:

\[
\Pi_i^d = n_{\beta} \gamma (2b-c) = (n-n_{\alpha}) \gamma (2b-c).
\]

Hence \( \Pi_i(s) > \Pi_i^d \) if and only if

\[
n_{\alpha} > (\gamma(2b-c))/((1/2)(2d-c)+\gamma(2b-c))n + (1/2)(2d-c)/(1/2)(2d-c)+\gamma(2b-c)),
\]

which simplifies to \( n_{\alpha}/n > [(\gamma(2b-c))/((1/2)(2d-c)+\gamma(2b-c))] \).

Second, we look the condition such that an agent \( j \in N_{\beta} \) does not have incentives to deviate.

The payoff if agent \( j \) follows the strategy \( s \in S_j \) is:

\[
\Pi_j(s) = ((n_{\beta}-1)/2) \gamma (2b-c)+((n_{\beta}-1)/2)(1-\gamma)(2b-c) = (n_{\beta}-1)(1/2)(2b-c) = (n-n_{\alpha}-1)(1/2)(2b-c).
\]

The payoff obtained if she deviates is:

\[
\Pi_j^d = n_{\alpha} \gamma (2d-c).
\]
Hence $\Pi(s) > \Pi^d$ if and only if

$$n_\alpha < \frac{(1/2)(2b-c)/(\gamma(2d-c)+(1/2)(2b-c))}n - \frac{(1/2)(2b-c)/(\gamma(2d-c)+(1/2)(2b-c))}n,$$

which simplifies to $n_\alpha/n < \frac{(1/2)(2b-c)/(\gamma(2d-c)+(1/2)(2b-c))}$. This completes the proof of the first part of the proposition.

**Second step: necessary condition**

To complete the proof we claim that if a strategy $s' \in S$ which provides outcome type III is an equilibrium, then any strategy $s \in S_3$ is an equilibrium. To see this note that conditions (III.i), (III.ii) and (III.iii) are necessary to obtain this outcome and to have subgame perfection, respectively. Hence, $s'$ must satisfy (III.i-iii). Note that conditions (III.iv) and (III.v) make that any agent who deviates obtains the worst possible outcome (since they become proposers in all their links). Thus, assume conditions (III.iv-v) are also satisfied by $s'$. Clearly the payoff attained by any agent in the set $N_\alpha$ who deviates to choose action $\beta$ is the same (because any agent who deviates becomes the proposer in all her links). Differently, the payoff an agent in the set $N_\alpha$ obtains from following the strategy is decreasing in the number of links he propose. Thus, the critical $\alpha$-player is the one who is proposing a higher number of links. The same argument holds for players in the set $N_\beta$. Any strategy in $S_3$ is characterized by (III.vi): $\sum_{k \in N_\alpha(i)} g_k(a) = (n_\alpha-1)/2 \text{ \forall } i \in N_\alpha$ and $\sum_{i \in N_\beta(j)} g_i(a) = (n_\beta-1)/2 \text{ \forall } j \in N_\beta$, i.e. each player is proposing half of the links that she finally forms. Hence the incentives to deviate are the same across all players and weakly lower than the incentives in strategy $s'$. This completes the proof.

**Proof of Proposition 5**

To show this result, we define a particular set of strategy profiles $S_\beta \subset S$, which provide the type IV outcome. We then show that, if $2b > e+f > c > d$, any strategy $s \in S_\beta$ is an equilibrium if and only if one of the conditions (i)-(ii) in the proposition holds. Hence, we prove that this represents a sufficient condition.

We define the set of strategy profiles $S_\beta \subset S$ such that $s = \{a, \{g(a)\}_{a \in A}^\beta \} \in S_\beta$ if and only if the following conditions (i)-(vi) hold:

**Proposition 5**

(i) $a = \alpha, \forall i \in N_\beta \subset N$

(ii) $a = \beta, \forall j \in N_\beta = \emptyset \setminus N_\alpha$

(iii) $g_i(a) + g_j(a) = 1 \text{ \forall } (i,j) \subset N \text{ and } \forall a \in A^\beta$

(iv) $g_i(a_i = \alpha, a_j = \beta) = 1 \text{ if } i \in N_\alpha \text{ and } j \in N_\beta$
Nevertheless, it is not proved that this assumption allows for an equilibrium in the widest parameter range. Thus, to show this result, we define a particular set of strategy profiles $S$. We then show that, if $2b > e + f > c > d$, the following conditions (i)-(vi) hold: 

\[(IV. i)\] $\forall i \in N_\alpha \exists j \in N_\beta : g_i(a_i, a_j) = 1$ if $i \in N_\alpha$ and $j \in N_\beta$

\[(IV. ii)\] $\forall i \in N_\beta \exists j \in N_\alpha : g_i(a_i, a_j) = 1$ if $i \in N_\beta$ and $j \in N_\alpha$

\[(IV. iii)\] $\sum_{k \in N_\alpha \setminus \{j\}} g_k(\hat{a}) = (n_\alpha - 1)/2$ $\forall i \in N_\alpha$ and $\sum_{l \in N_\beta \setminus \{j\}} g_l(\hat{a}) = (n_\beta - 1)/2$ $\forall j \in N_\beta$

where $n_\alpha = |N_\alpha|$ and $n_\beta = |N_\beta| = n - n_\alpha$. In words, a strategy belongs to $S$ if there is a set of players $N_\alpha$ who choose the efficient action and a set $N_\beta = N \setminus N_\alpha$ of players who choose the risk-dominant action (\(IV.i\))-\(IV.ii\), and, for any possible pattern of actions each bilateral link is proposed by one and only one of the players involved (\(IV.iii\)). Moreover we impose that, any player who deviates to choose a different action becomes the proposer of any possible profitable link she may form (\(IV.iv\))-\(IV.vi\). Additionally, in any strategy $s \in S$, in the equilibrium path, each player proposes half of the links she finally forms with players choosing the same action as her (\(IV.vii\)). These two conditions are imposed in order to allow for an equilibrium for the widest parameter range. Finally we comment on condition (\(IV.iv\)) which prescribes that in the equilibrium path each player in the set $N_\alpha$ proposes links to all players in the set $N_\beta$. This condition is assumed to make the incentives to deviate of the $\beta$-players as low as possible.\((26)\)

Assume $\min\{2b, e + f\} > c > d$. Let player $i$ be the proposer of the link $ij$. Then from Section 3 we know that the following net payoffs arise:

- If $(a, a) = (\alpha, \alpha)$ then $(\hat{R}_i, \hat{R}_j) = (\gamma(2d-c), (1-\gamma)(2d-c))$
- If $(a, a) = (\beta, \beta)$ then $(\hat{R}_i, \hat{R}_j) = (\gamma(2b-c), (1-\gamma)(2b-c))$
- If $(a, a) = (\alpha, \beta)$ then $(\hat{R}_i, \hat{R}_j) = (\gamma(e+f-c), (1-\gamma)(e+f-c))$
- If $(a, a) = (\beta, \alpha)$ then $(\hat{R}_i, \hat{R}_j) = (\gamma(e+f-c), (1-\gamma)(e+f-c))$

Moreover assume $2b > e + f$. First, we look for the condition such that an agent $i \in N_\alpha$ does not want to deviate.

The payoff agent $i$ obtains if she follows the strategy $s \in S$ is:

$\Pi_i(s) = (n_\alpha - 1)(1/2)(2d-c) + (n-n_\alpha) \gamma (e+f-c)$.

The payoff obtained if she deviates is:

\[(26)\text{Nevertheless, it is not proved that this assumption allows for an equilibrium in the widest parameter range. Thus, in the statement of the proposition we just provide sufficient conditions. In any case, the equilibrium analysis suggests that } \beta \text{-players are more critical (for deviating) than } \alpha \text{-players, which leads us to assume this condition.}\]
\[ \Pi_j^d = (n_\alpha - 1) \gamma (e+f-c) + (n-n_\alpha) \gamma (2b-c). \]

Hence \( \Pi_j(s) > \Pi_j^d \) if and only if

\[ n_\alpha > \frac{(\gamma(2b-(e+f))/(\gamma(2b-(e+f))+(1/2)(2d-c)-\gamma(e+f-c))) \times n + +((1/2)(2d-c)-\gamma(e+f-c))/(\gamma(2b-(e+f))+(1/2)(2d-c)-\gamma(e+f-c))}{(n-n_\alpha - 1) \gamma (e+f-c) + (n-n_\alpha) \gamma (2d-c)}, \]

which simplifies to \( n_\alpha/n > \left[ \frac{\gamma(2b-(e+f))/(\gamma(2b-(e+f)+c+d-(c/2))]}{\gamma(2b-(e+f)-(c/2))}. \]

We now look for the conditions such that a player \( j \in N_\beta \) does not have incentives to deviate.

The payoff agent \( j \) obtains if she follows the strategy \( s \in S_4 \) is:

\[ \Pi_j(s) = (n-n_\alpha - 1)(1/2)(2b-c) + n_\alpha (1-\gamma)(e+f-c). \]

The payoff obtained if she deviates is:

\[ \Pi_j^d = (n-n_\alpha - 1) \gamma (e+f-c) + (n-n_\alpha) \gamma (2d-c). \]

Hence \( \Pi_j(s) > \Pi_j^d \) if and only if

\[ n_\alpha < \frac{((1/2)(2d-c)-\gamma(e+f-c))/((1/2)(2b-c)-\gamma(e+f-c)+\gamma(2d-c)-(1-\gamma)(e+f-c))) (n-1)}{n_\alpha/n > \left[ \frac{\gamma(2b-(e+f))/(\gamma(2b-(e+f)+c+d-(c/2))]}{\gamma(2b-(e+f)-(c/2))}. \]

Note that, if \( \gamma<(e+f-c)/(2d-c+e+f-c) \), then \( \gamma(2d-c)-(1-\gamma)(e+f-c)<0 \), hence in this case the condition for the \( \beta \)-player is always satisfied for any \( n_\alpha/n \).

Differently, if \( \gamma>(e+f-c)/(2d-c+e+f-c) \) the condition for the \( \beta \)-player is binding, and can be simplified to \( n_\alpha/n < \left[ ((1/2)(2b-c)-\gamma(e+f-c))/(1/2)(2b-c)+\gamma(2d-c)-(e+f-c)) \].

This completes the proof. \( \blacksquare \)

**Proof of Proposition 6**

The proof has two steps. In the first step we show that, if \( e+f>2b>c>d \), any strategy \( s \in S_4 \) (this set is defined in the proof of proposition 5 and represent a particular set of strategy profiles which provide the outcome type IV) is an equilibrium if and only if one of the conditions (i)-(iv) in the proposition holds. Hence, we prove that this represents a sufficient condition. In the second step we show that if any strategy \( s' \in S \) resulting in outcome type IV is an equilibrium, then any strategy in the set \( S_4 \) constitute an equilibrium. Thus, we prove that the proposition also provides necessary conditions.

**First step: sufficient condition**

Assume \( e+f>2b>c>d \). We look for the conditions such that neither \( \alpha \)-players nor \( \beta \)-players have incentives to deviate from \( s \in S_4 \) (note that this set of strategies is defined in the proof of proposition 5 by conditions (IV.i)-(IV.vii)). First, we look
for the condition such that an agent \(i \in N_\alpha\) does not want to deviate. The payoffs of agent \(i\) if she follows the strategy \(s \in S_i\), \(\Pi_i(s)\), and if she deviates to choose action \(\beta\), \(\Pi_i^d\), were already obtained in the proof of proposition 5. We note that, when \(e+f>2b\):

\[
\Pi_i(s) = (n_\alpha - 1)\gamma(e+f-c) + (n-n_\alpha)\gamma(e+f-c) > (n_\alpha - 1)\gamma(e+f-c) + (n-n_\alpha)\gamma(2b-c).
\]

Then, in this case an \(\alpha\)-player does not have incentives to deviate, whatever the cardinality of the set \(N_\alpha\).

We now investigate the conditions for an agent \(j \in N_\beta\).

From the previous proposition, we already got this condition, which may be rewritten as:

\[
n_\alpha [\gamma(e+f-c)-(1/2)(2d-c)+(1-\gamma)(e+f-c)-\gamma(2d-c)] > (n-1)[\gamma(e+f-c)-(1/2)(2d-c)].
\]

Let us denote:

\[
\Lambda = \gamma(e+f-c)-(1/2)(2d-c)
\]

\[
\Omega = \gamma(e+f-c)-(1/2)(2d-c)+(1-\gamma)(e+f-c)-\gamma(2d-c)
\]

Then the condition for the \(\beta\)-player becomes:

\[
\Omega n_\alpha > \Lambda (n-1) \quad (5)
\]

We note that:

\([\Lambda > 0 \iff \gamma > (2b-c)/(2(e+f-c)]) \text{ and } [\Omega > 0 \iff \gamma < (2(e+f-b)-c)/(2(2d-c))]\]

We now analyse the possibilities [1]-[4]:

[1] \(\gamma < \min\{(2b-c)/(2(e+f-c)), (2(e+f-b)-c)/(2(2d-c))\}\). In this case \(\Lambda<0\) and \(\Omega>0\). So the condition for the \(\beta\)-player is satisfied for any cardinality of the set \(N_\alpha\).

[2] \((2b-c)/(2(e+f-c)) > \gamma > (2(e+f-b)-c)/(2(2d-c))\). In this case \(\Lambda<0\) and \(\Omega<0\). Hence condition (5) becomes:

\[
[(1/2)(2b-c)-\gamma(e+f-c)+\gamma(2d-c)-(1-\gamma)(e+f-c)] n_\alpha < [(1/2)(2b-c)-\gamma(e+f-c)](n-1)
\]

Note that, if \(\gamma < (e+f-c)/(2d-c+e+f-c)\), then \(\gamma(2d-c)-(1-\gamma)(e+f-c) < 0\), hence in this case the condition for the \(\beta\)-player is always satisfied for any \(n_\alpha < n\).

Differently, if \(\gamma > (e+f-c)/(2d-c+e+f-c)\), the condition for the \(\beta\)-player is binding, and becomes:

\[
n_\alpha < \left(\frac{(1/2)(2b-c)-\gamma(e+f-c)}{(1/2)(2b-c)+\gamma(2d-c)-(e+f-c)}\right) (n-1)
\]
[3] \( \gamma > \max\{2b-\alpha)/(2(e+f-\alpha),(2(e+f-b)-\alpha)/(2d-\alpha)\} \). In this case \( \Lambda > 0 \) and \( \Omega > 0 \). Hence condition (5) becomes:

\[
\gamma(e+f-\alpha)-(1/2)(2b-\alpha)+(1-\gamma)(e+f-\alpha)-(2d-\alpha) > \gamma(e+f-\alpha)-(1/2)(2b-\alpha) \quad (n-1)
\]

Note that, if \( \gamma > (e+f-\alpha)/(2d+c+e+f-\alpha) \), then \((1-\gamma)(e+f-\alpha)-(2d-\alpha) < 0 \), hence there is no equilibrium (an equilibrium would require \( n_\alpha > n \)).

Differently, if \( \gamma < (e+f-\alpha)/(2d-c+e+f-\alpha) \), the condition for the \( \beta \)-player becomes:

\[
n_\alpha > (\gamma(e+f-\alpha)-(1/2)(2b-\alpha))/((e+f-\alpha)-(1/2)(2b-\alpha)-(2d-\alpha)) \quad (n-1)
\]

[4] \((2(e+f-b)-\alpha)/(2d-\alpha) > \gamma > (2b-\alpha)/(2(e+f-\alpha)) \). In this case \( \Lambda > 0 \) and \( \Omega < 0 \). Hence condition (5) is never satisfied.

Hence, when \( e+f > 2b > c > d \), we have shown that a strategy \( s \in S_4 \) is an equilibrium if and only if condition (i), (ii), (iii) or (iv) in the proposition holds.

**Second step: necessary condition**

We now claim that the fact that one of these conditions (i)-(iv) is satisfied is necessary for a type IV outcome to be sustained as equilibrium. To this aim, we claim that, if a strategy \( s' \in S \) which provides a type IV outcome is an equilibrium, then any strategy \( s \in S_4 \) is an equilibrium. To see this note that conditions (IV.i), (IV.ii) and (IV.iii) are necessary to obtain this outcome and to have subgame perfection, respectively. Hence, \( s' \) must satisfy (IV.i-iii). Note that conditions (IV.v) and (IV.vi) make that any agent who deviates obtains the worst possible outcome (since they become proposers in all their links). Thus, assume conditions (IV.v) and (IV.vi) are also satisfied by \( s' \). Moreover assume that \( s \) prescribes that each \( \alpha \)-player proposes, in the equilibrium path links to all \( \beta \)-players (condition (IV.iv)).\(^{27}\) Clearly the payoff attained by any agent in the set \( N_\alpha \) who deviates to choose action \( \beta \) is the same (because any agent who deviates becomes the proposer in all her links). Differently the payoff an agent in the set \( N_\alpha \) obtains from following the strategy is decreasing in the number of links he propose to players choosing the same action as her. Hence, the critical \( \alpha \)-player is that one who is proposing a higher number of links to players in the set \( N_\alpha \). The same argument holds for players in the set \( N_\beta \). Any strategy in \( S_4 \) is characterized by (IV.vii): \( \Sigma_{k \in N_{\alpha \smallsetminus \alpha(i)}} g_{\alpha}(\hat{a}) = (n_\alpha-1)/2 \forall \in N_\alpha \) and \( \Sigma_{k \in N_{\beta \smallsetminus \alpha(i)}} g_{\beta}(\hat{a}) = (n_\beta-1)/2 \forall \in N_\beta \), i.e. each player is proposing half of the links to players in her set. Hence, the incentives to deviate are the same across all players choosing the same action, and weakly lower than the incentives in strategy \( s \). Finally

\(^{27}\) We will show below that this is a necessary condition for this outcome to be supported in equilibrium for the widest parameter range.
we recall that, from the analysis above, that given $s \in S_1$, $\alpha$-players do not have incentives to deviate, even if they are proposing all the links to players in $N_1$. Hence, the critical player is in the set $N_2$, and condition (V.iv) becomes necessary in order to get this equilibrium outcome for the widest parameter range. This completes the proof.

Proof of Proposition 7

The proof has two steps. In the first step we define a particular set of strategy profiles $S_5 \subseteq S$, which provide the type $V$ outcome. We then show that, if $e+f>c=\max\{2b,d\}$, any strategy $s \in S_5$ is an equilibrium if and only if the conditions stated in the proposition hold. Hence, we prove that this represents a sufficient condition. In the second step, we show that if any strategy $s' \in S$ resulting in outcome type $V$ is an equilibrium, then any strategy in the set $S_5$ constitute an equilibrium. Thus, we prove that the proposition also provides the necessary condition.

First step: sufficient condition

Let us define the set of strategy profiles $S_5 \subseteq S$ such that $s = \{\hat{a}, \{g(\alpha)\}_{\alpha \in A^n}\} \in S_5$ if and only if the following conditions (V.i)-(V.vii) hold:

(V.i) $\hat{a}_i = \alpha$, $\forall i \in N_\alpha \subseteq N$

(V.ii) $\hat{a}_j = \beta$, $\forall j \in N_\beta = N \setminus N_\alpha$

(V.iii) $g_i(\alpha) + g_j(\beta) = \begin{cases} 1 & \text{if } \pi(a_i, a_j) + \pi(a_j, a_i) \geq c \\ 0 & \text{otherwise} \end{cases} \quad \forall \{i,j\} \subseteq N$ and $\forall a \in A^n$

(V.iv) $g_i(\alpha) + g_j(\beta) = 1$ if $i \in N_\alpha$ and $j \in N_\beta$

(V.v) $g_i(\alpha) + g_j(\beta) = 1$ if $i \in N_\beta$ and $j \in N_\alpha$

(V.vi) $g_i(\alpha) + g_j(\beta) = 1$ if $i \in N_\alpha$ and $j \in N_\beta$

(V.vii) $\sum_{k \in N_\alpha \setminus \{i\}} g_k(\hat{a}) = (n_\alpha - 1)/2 \quad \forall i \in N_\alpha$

In words, a strategy belongs to $S_5$ if there is a set of players $N_\alpha$, who choose the efficient action, and a set $N_\beta = N \setminus N_\alpha$, of players who choose the risk-dominant action (V.i)-(V.ii), and, whenever the aggregate gross payoff from a link exceeds the cost, the link is proposed by one and only one of the players involved (V.iii). Moreover we impose that, in the equilibrium path, each $\alpha$-player proposes links to all players choosing action $\beta$ (V.iv), and any player who deviates to choose a
different action to that one prescribed by the strategy becomes the proposer of any possible (profitable) link she may form (V.v)-(V.vi). Additionally, in any strategy \( s \in S_5 \), in the equilibrium path, each \( \alpha \)-player proposes half of the links she finally forms with players choosing the same action (V.vii). These conditions are imposed in order to attain the strategy profiles which support the outcome type V for the widest parameter range.

Assume \( e+f>c>\max\{2b,d\} \). Let player \( i \) be the proposer of the link \( ij \). Then from Section 3 we have:

- If \((a_i,a_j) = (\alpha,\alpha)\) then \((\pi^i,\pi^j) = (\gamma(2d-c),(1-\gamma)(2d-c))\)
- If \((a_i,a_j) = (\beta,\beta)\) then \((\pi^i,\pi^j) = (0,0)\)
- If \((a_i,a_j) = (\alpha,\beta)\) then \((\pi^i,\pi^j) = (\gamma(e+f-c),(1-\gamma)(e+f-c))\)
- If \((a_i,a_j) = (\beta,\alpha)\) then \((\pi^i,\pi^j) = (\gamma(e+f-c),(1-\gamma)(e+f-c))\)

First we look for the condition such that an agent \( i \in N_\alpha \) does not want to deviate. The payoff agent \( i \) obtains if she follows the strategy \( s \in S_5 \) is:

\[
\Pi^i(s) = ((n_\alpha-1)/2) \gamma (2d-c) + ((n_\alpha-1)/2) (1-\gamma) + (n-n_\alpha)(e+f-c) = (n_\alpha-1)(1/2) (1-\gamma) + (n-n_\alpha)(e+f-c)
\]

The payoff obtained if she deviates is:

\[
\Pi^i_d = (n_\alpha-1) \gamma (e+f-c).
\]

Clearly \( \Pi^i(s) > \Pi^i_d \) and hence no agent in the set \( N_\alpha \) has incentives to deviate.

We now look for the conditions such that a player \( j \in N_\beta \) does not have incentives to deviate.

The payoff agent \( j \) obtains if she follows the strategy \( s \in S_5 \) is:

\[
\Pi^j(s) = n_\alpha (1-\gamma) (e+f-c).
\]

The payoff obtained if she deviates is:

\[
\Pi^j_d = n_\alpha \gamma (2d-c) + (n-n_\alpha-1) \gamma (e+f-c).
\]

Clearly, if \( \gamma>(e+f-c)/(2d-c) \), i.e. \( \gamma(2d-c)>e+f-c \), then \( \Pi^j(s) > \Pi^j_d \). Hence, a necessary condition for an equilibrium in this case is \( \gamma<(e+f-c)/(2d-c) \). Then, note that \( \Pi^j(s) > \Pi^j_d \) if and only if \( n_\alpha>(\gamma(e+f-c)/(e+f-c-\gamma(2d-c)))(n-1) \), but this condition is only possible when \( (e+f-c-\gamma(2d-c))>\gamma(e+f-c) \), i.e. \( \gamma<(e+f-c)/(2d-c+e+f-c) \). This proves the first part of the proposition.
Second step: necessary condition

To complete the proof we claim that if a strategy $s' \in S$ which provides a type V outcome is an equilibrium, then any strategy $s \in S_\beta$ is an equilibrium. To see this note that conditions (V.i), (V.ii) and (V.iii) are necessary to obtain this outcome and to have subgame perfection, respectively. Hence, $s'$ must satisfy (V.i-iii). Note that we showed that, in this case, $\alpha$-players do not have incentives to deviate even if in the equilibrium path they propose links to all $\beta$-players (represented by condition (V.iv)), and, conditions (V.v) and (V.vi) make that any agent who deviates obtains the worst possible outcome (since they become proposers in all their links). Thus, assume conditions (V.iv-vi) are also satisfied by $s'$. Clearly, the payoff attained by any agent in the set $N_\alpha$ who deviates to choose action $\beta$ is the same (because any agent who deviates becomes the proposer in all her links). Differently, the payoff an agent in the set $N_\alpha$ obtains from following the strategy is decreasing in the number of links he propose to players choosing the same action. Hence the critical $\alpha$-player is that one who is proposing a higher number of links. Differently all $\beta$-players have the same incentives to deviate, which are minimized by means of the previous conditions. Any strategy in $S_\beta$ is characterized by (V.vii): $\sum_{k \in N_\alpha \setminus \{i\}} g_{jk}(\bar{a}) = (n_\alpha - 1)/2$ $\forall \in N_\alpha$, i.e. each $\alpha$-player is proposing half of the links that she finally forms with players in $N_\alpha$. Hence, the incentives to deviate are the same across all $\alpha$-players and weakly lower than the incentives in strategy $s'$. This completes the proof. ■